Journal of Engineering Physics and Thermophysics, Vol. 79, No. 5, 2006

# MOMENTUM EQUATION AND SIMPLE WAVES IN AN ELASTIC PIPELINE

**A. N. Volobuev and A. P. Tolstonogov**

UDC 532.5.52

*A comparative analysis of momentum equations for an ideal-liquid flow in rigid and elastic pipelines has been performed. A fundamental difference of these equations has been shown using individual examples. The influence of the character of the boundary has been analyzed. A general solution analogous to the Riemann solution for propagation of a simple nonlinear wave in an elastic pipeline has been obtained. It has been shown that the wave shape changes for any velocity of the liquid in the pipeline and a shock wave occurs. The formation length of the shock wave has been evaluated.*

In calculating the flow of an ideal liquid in a rigid pipeline, one uses the Euler equation

$$-\frac{dP}{dX}=\rho\frac{dV}{dt}\,. \tag{1}$$

In an elastic pipeline [1], the momentum equation for such a flow changes only slightly:

$$-\frac{\partial\,(PS)}{S\partial X}=\rho\frac{dV}{dt}\,. \tag{2}$$

The momentum equation closest to (2) for an ideal-liquid flow in an elastic pipeline has been derived in [2]:

$$-\frac{\partial\,(PS)}{\partial X}=\frac{d\,(S\rho V)}{dt}\,. \tag{3}$$

Let us assume that the density of the liquid ρ is a constant. Then (3) will take the form

$$-\frac{\partial\,(PS)}{\partial X}=\rho\left[S\frac{\partial V}{\partial t}+SV\frac{\partial V}{\partial X}+V\left(\frac{\partial S}{\partial t}+V\frac{\partial S}{\partial X}\right)\right]. \tag{4}$$

We transform Eq. (4), using the continuity equation [3]

$$\frac{\partial S}{\partial t}+\frac{\partial\,(SV)}{\partial X}=0\,. \tag{5}$$

It is easily shown that the sum of the last three terms on the right-hand side of (4) is equal to zero. Unlike (4), where the partial derivative $\frac{\partial V}{\partial t}$ is left on the right-hand side, the total derivative $\frac{\partial V}{\partial t}$ is written in Eq. (2).

**Application of the Momentum Equation to the Flow in an Elastic Pipeline.** The interrelation of Eqs. (1) and (2) is of great general physical importance, since Eq. (2) represents at first glance a certain generalization of Eq. (1).

In considering the first model problem, we assume that $S$ is a constant. Taking $S$ outside the sign of the derivative with respect to $X$ in Eq. (2), we obtain Eq. (1). As will be shown in the following discussion, such a consideration is incorrect and Eq. (2) is not a generalization of (1).

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Akad. S. P. Korolev Samara State Aerospace University, 34 Moskovskoe Shosse, Samara, 443086, Russia; email: volobuev@samaramail.ru. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 79, No. 5, pp. 147–151, September–October, 2006. Original article submitted December 20, 2004; revision submitted February 16, 2006.

1062-0125/06/7905-0993 

We consider the second model problem. Let us assume that an ideal liquid flows in a rigid pipeline in which the pressure $P$ is constant along the length $X$ and does not depend on the time $t$, whereas the cross-sectional area of the pipeline $S$ depends on the coordinate $X$ and is not a function of the time $t$. We take the pressure $P$ outside the sign of the derivative with respect to $X$ in Eq. (2); as a result, we obtain

$$-P\frac{\partial S}{S\partial X}=\rho\frac{dV}{dt}. \tag{6}$$

The continuity equation (5) is simplified and we can write it as $SV$ = const. What this means is that if the area is not a function of time, the velocity $V$ does not depend on time, too, but the dependence of the velocity on $X$ is preserved. Consequently, the momentum equation (6) acquires the form

$$-P\frac{\partial S}{S\partial X}=\rho V\frac{dV}{dX}. \tag{7}$$

Equation (7) reflects a certain stationary ideal-liquid flow, since its parameters — pressure, velocity, and cross-sectional area of the pipeline — remain constant with time. If we use the continuity equation

$$-V\frac{\partial S}{\partial X}=S\frac{\partial V}{\partial X}, \tag{8}$$

for such a flow, we obtain an expression for the pressure $P = \rho V^2$. The last relation is incorrect, since we have the quantity independent of the coordinate $X$ on the left and the velocity function dependent on it on the right. The explanation for this "paradox" is as follows. Nothing has been said about the character of the boundary until the present time. The boundary can be absolutely rigid and can itself form the liquid flow. The Euler equation (1) must be used in such problems. However, the boundary can be elastic to such an extent that the liquid flow itself forms the boundary. Here Eq. (2) is expediently used.

In the second model problem, we used Eq. (2) for an elastic pipeline to calculate a rigid pipeline; this led us to a paradoxical result, i.e., we actually solved a problem in which the flow forms the boundary in such a manner that the area $S$ does not depend on time $t$ and is a function of the coordinate $X$. But we cannot have such a flow in an elastic pipeline for the pressure independent of the time $t$ and the coordinate $X$. Flow in an elastic pipeline is a fundamentally nonstationary process. The pressure in the flow and the cross-sectional area of the pipeline cannot but depend on the time $t$. Indeed, any random increase in the flow velocity at any point of an elastic pipeline leads to a decrease in the flow pressure in it due to the "Bernoulli effect." Due to the elasticity of the pipeline, its cross-sectional area begins to decrease, too, which makes the flow velocity even higher. A positive feedback leading to a collapse of the elastic pipeline develops. However the pipeline diverges with a certain flow rate of the liquid. The process recurs periodically. We have self-oscillations of the pipeline wall and of the liquid flow in the pipeline, i.e., the so-called flow–wall instability.

Consequently, it is impossible to take the area $S$ in Eq. (2) beyond the sign of the derivative with respect to $X$ (just as in the first model problem). Even if the cross-sectional area of the pipeline did not depend on the coordinate $X$ in the absence of a liquid flow, we would observe self-oscillating flow, or flow flutter [4], upon the development of the flow.

Equation (2) virtually never becomes Eq. (1), since, unlike (1), it describes a fundamentally nonstationary process in which the flow is involved in the formation of the boundary. In solving any hydrodynamic problem, we should primarily elucidate the character of the boundary (whether it is rigid or elastic) and subsequently use Eqs. (1) or (2). The latter is the same independent hydrodynamic equation as (1), although it is clear that both these equations reflect the second Newton law for an ideal liquid.

**Simple Waves in an Elastic Pipeline.** Solitary waves can occur in elastic pipelines [5, 6]. We consider this phenomenon using simple waves as an example [3].

Following [3], we find the solution of the system of equations (2) and (5). For this purpose we rewrite (2) in the following form:

$$\frac{\partial V}{\partial t}+\left(V+\frac{1}{\rho}\frac{\partial\,(PS)}{S\partial V}\right)\frac{\partial V}{\partial X}=0\,, \qquad (9)$$

whence we can obtain

$$\left(V+\frac{1}{\rho}\frac{\partial\,(PS)}{S\partial V}\right)=-\frac{\partial V}{\partial t}\bigg/\frac{\partial V}{\partial X}=\frac{\partial X}{\partial t}\,. \qquad (10)$$

We carry out analogous transformations in the continuity equation (5):

$$\frac{\partial S}{\partial t}+\left(V+S\frac{\partial V}{\partial S}\right)\frac{\partial S}{\partial X}=0\,. \qquad (11)$$

Consequently, we have

$$V+S\frac{\partial V}{\partial S}=-\frac{\partial S}{\partial t}\bigg/\frac{\partial S}{\partial X}=\frac{\partial X}{\partial t}\,. \qquad (12)$$

Equating (10) and (12), we find

$$\frac{1}{\rho}\frac{\partial\,(PS)}{S\partial V}=S\frac{\partial V}{\partial S}\,. \qquad (13)$$

Using the equality $c=\sqrt{\frac{\partial(PS)}{\rho\partial S}}$ [1], we transform (13) into the relation

$$\frac{1}{\rho}\frac{\partial\,(PS)}{S\partial V}=\frac{1}{\rho}\frac{\partial\,(PS)}{S\partial S}\frac{\partial S}{\partial V}=\frac{c^2}{S}\frac{\partial S}{\partial V}=S\frac{\partial V}{\partial S} \qquad (14)$$

or

$$\frac{1}{\rho}\frac{\partial\,(PS)}{S\partial V}=S\frac{\partial V}{\partial\,(PS)}\frac{\partial\,(PS)}{\partial S}=S\frac{\partial V}{\partial\,(PS)}\rho c^2\,. \qquad (15)$$

Then, from (14) and (15), we have

$$V==\pm\int\frac{c}{S}\,dS=\pm\int\frac{d\,(PS)}{\rho c S}\,. \qquad (16)$$

The solution found for the system of nonlinear hydrodynamic equations (2) and (5) is referred to as the "Riemann solution." It is satisfied by nonlinear waves propagating in positive and negative directions of the $X$ axis; such waves are often called simple waves.

If a solitary wave of small amplitude propagates in an elastic pipeline, the cross-sectional area of the pipeline can be represented as $S = S_0+\Delta S$. On the basis of (16) and allowing for the condition $S_0 >> \Delta S$, we find

$$V=c\int\frac{d\,(\Delta S)}{S_0}=c\frac{\Delta S}{S_0}\,. \qquad (17)$$

Using the Hooke law in the form $P = D\frac{\Delta S}{S_0}$ and the formula $D = \rho c^2$ [6], we determine the longitudinal velocity of the liquid as $V = \frac{c}{D}P = \frac{P}{\rho c}$. The well-known relation [3] has been obtained, which points to the correctness

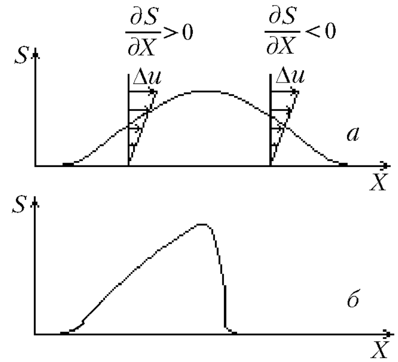


Fig. 1. Cross-sectional area of an elastic pipeline in a simple nonlinear wave vs. coordinate $X$ (a) and the appearance of a discontinuity surface in the wave (b) in the case of occurrence of a shock wave in the elastic pipeline.

of the transformations carried out. From the above analysis, it is clear that a "plus" sign must be taken in (16) in the problem in question. On the basis of (10) and (16), we find

$$\frac{\partial X}{\partial t} = V + \frac{1}{\rho} \frac{\partial (PS)}{S \partial V} = V + c(V) . \tag{18}$$

We assume that the velocities $V$ and $c$ weakly depend on time, i.e., a nearly stationary wave propagates in the pipeline. In this case, the solution of Eq. (18) will take the form

$$X = [V \pm c(V)]\, t + f(V) . \tag{19}$$

It is easily shown that the term $f(V)$ is equal to 0 for the self-similar solution in the coordinate $X/t$ [3].

**Occurrence of Shock Waves in an Elastic Pipeline.** Formulas (16) and (19) determine, in an implicit manner, the liquid velocity in an elastic pipeline as a function of the coordinate $X$ and the time $t$. A local divergence of the pipeline results in a certain wave profile [3]. It follows from (19) that points in the wave move with a velocity

$$u = V + c(V) . \tag{20}$$

The stationary solution (19) does not hold in propagation of a simple wave [3, 7]. The upper points of the wave move with a velocity $\Delta u$ higher than the lower points (Fig. 1a). Different velocities of motion of the points in the wave lead to a change in its shape (Fig. 1b). The process of propagation of the wave becomes nonstationary, which is characteristic of an elastic pipeline, i.e., the solution (19) does not hold. The wave shape changes with time and a discontinuity surface, i.e., a shock wave, is formed, which is a distinctive feature of simple nonlinear waves [7].

To analyze the process of change in the shape of a solitary wave we rewrite Eq. (9) with account for (18):

$$\frac{\partial V}{\partial t} + (V + c) \frac{\partial V}{\partial X} = 0 . \tag{21}$$

According to [7], a particular solution of the nonlinear wave equation (21) with the boundary condition $V = V(t)$ for $X = 0$ has the form

$$V = V\left(t - \frac{X}{V + c}\right). \tag{22}$$

We perform further analysis of the occurrence of a shock front in a simple nonlinear wave, mainly following [7], where such an analysis has been made for a sound wave in a gas. Taking into account that $V << c$, we transform the solution (22):

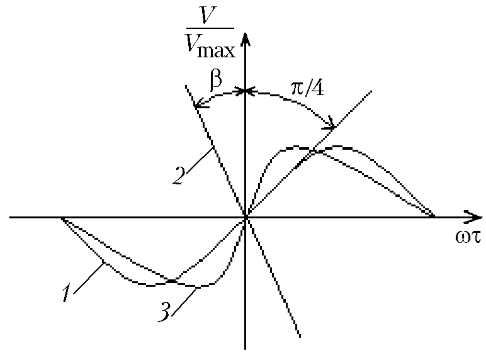


Fig. 2. Change in the shape of a shock wave in an elastic pipeline.

$$V = V\left(\tau + \frac{VX}{c^2}\right). \tag{23}$$

We analyze a change in the wave shape using the simplest sine wave as an example:

$$V = V_{\max} \sin\left[\omega\left(\tau + \frac{VX}{c^2}\right)\right]. \tag{24}$$

The dependence $V/V_{\max}$ = sin ($\omega\tau$) for $X$ = 0 is shown in Fig. 2, curve 1. A coordinate system moving with a velocity $c$ in the direction of propagation of the wave is used in this case. From (24), we find the value of the phase $\omega\tau$:

$$\omega\tau = \arcsin\left(\frac{V}{V_{\max}}\right) - \frac{\omega VX}{c^2} = \arcsin\left(\frac{V}{V_{\max}}\right) - z\frac{V}{V_{\max}} \approx \frac{V}{V_{\max}}(1 - z)\ . \tag{25}$$

Evaluation is performed accurate to $(V/V_{\max})^3$; consequently, we have

$$\frac{V}{V_{\max}} \approx \frac{\omega\tau}{1 - z}. \tag{26}$$

With growth in $X$ and for $z \to 1$, the quantity $V/V_{\max}$ tends to infinity, i.e., a discontinuity surface appears and a shock wave occurs. The increase in the steepness of the wave front is diagrammatically shown in Fig. 2. The quantity $z = \tan\,\beta$, on the basis of (25), represents a reciprocal slope of the straight line 2, whose equation is $V/V_{\max} = -\omega\tau/z$. When the slope β approaches π/4 with increase in $X$, the front of curve 3, which is the sum of sinusoid 1 and the straight line 2 for low $V/V_{\max}$ values, becomes vertical: a discontinuity surface appears in the wave. This phenomenon can be hindered by a small length of the pipeline, the viscosity of the liquid, the occurrence of bending moments in the pipeline walls, or the dispersion effects in the wave. In the last case, a stable soliton can occur [5–7]. However, in propagation of a nonlinear wave in an elastic pipeline, the leading front of the wave has a larger steepness than the trailing front.

The condition $z$ = 1 determines the minimum possible length of formation of a shock wave:

$$X_{\min} = \frac{c^2}{\omega V_{\max}} = \frac{\lambda}{2\pi M}\ . \tag{27}$$

In the absence of the liquid viscosity, a shock wave is formed for any nonzero liquid velocity and for any Mach number in a thin-walled elastic pipeline. For example, for a wavelength of $\lambda \approx 2.5$ m, liquid velocity $V_{\max}$ = 0.4 m/sec, and a velocity of propagation of the wave of $c$ = 8 m/sec (M = 0.05), the formation length of a shock wave is $X_{\min} \approx 8$ min.

## CONCLUSIONS

1. In solving the problem of liquid flow in an elastic pipeline, it is expedient to use a momentum equation in the form (2).

2. We can have shock waves in an elastic pipeline. The minimum possible length of formation of a shock wave is in direct proportion to the wave length and in inverse proportion to the Mach number.

## NOTATION

$c$, velocity of the pressure wave, m/sec; $D$, elasticity of the elastic-pipeline walls, $N/m^2$; $f(V)$, arbitrary velocity function, m; $M = V_{max}/c$, analog of the amplitude Mach number; $P$, pressure in the liquid flow, Pa; $S$, cross-sectional area of the pipeline, $m^2$; $S_0$, cross-sectional area of the pipeline outside the solitary wave, $m^2$; $t$, time, sec; $u$, velocity of movement of the wave's points, m/sec; $V$, longitudinal velocity of the liquid, m/sec; $V_{max}$, maximum longitudinal velocity in the wave, m/sec; $X$, longitudinal coordinate, m; $X_{min}$, minimum possible formation length of a shock wave, m; $z = \omega V_{max} X/c^2$, dimensionless combination; $\beta$, angle (slope), rad; $\Delta S$, increment in the cross-sectional area in the solitary wave, $m^2$; $\Delta u$, increment in the velocity of the wave on its crest, m/sec; $\lambda$, wavelength, m; $\rho$, density of the liquid, $kg/m^3$; $\tau = t - X/c$, phase combination, sec; $\omega$, cyclic frequency the wave, rad/sec. Subscripts: max and min, maximum and minimum values.

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